The Random Intercept Model

PSYC 575

August 6, 2020 (updated: 5 September 2021)

Week Learning Objectives

- Explain the components of a **random intercept model**
- Interpret **intraclass correlations**
- Use the **design effect** to decide whether MLM is needed
- Explain why ignoring clustering (e.g., regression) leads to inflated chances of Type I errors
- Describe how MLM **pools information** to obtain more stable inferences of groups

Data 1982 High School and Beyond Survey[1]

- 7,185 students (10-12th graders) from 160 schools (90 public and 70 Catholic)

- Level 1: Student
  - `id`: group identifier
  - `minority`: (1 = minority, 0 = not)
  - `female`: 1 = female, 0 = male
  - `ses`
  - `mathach`: Mathematics achievement

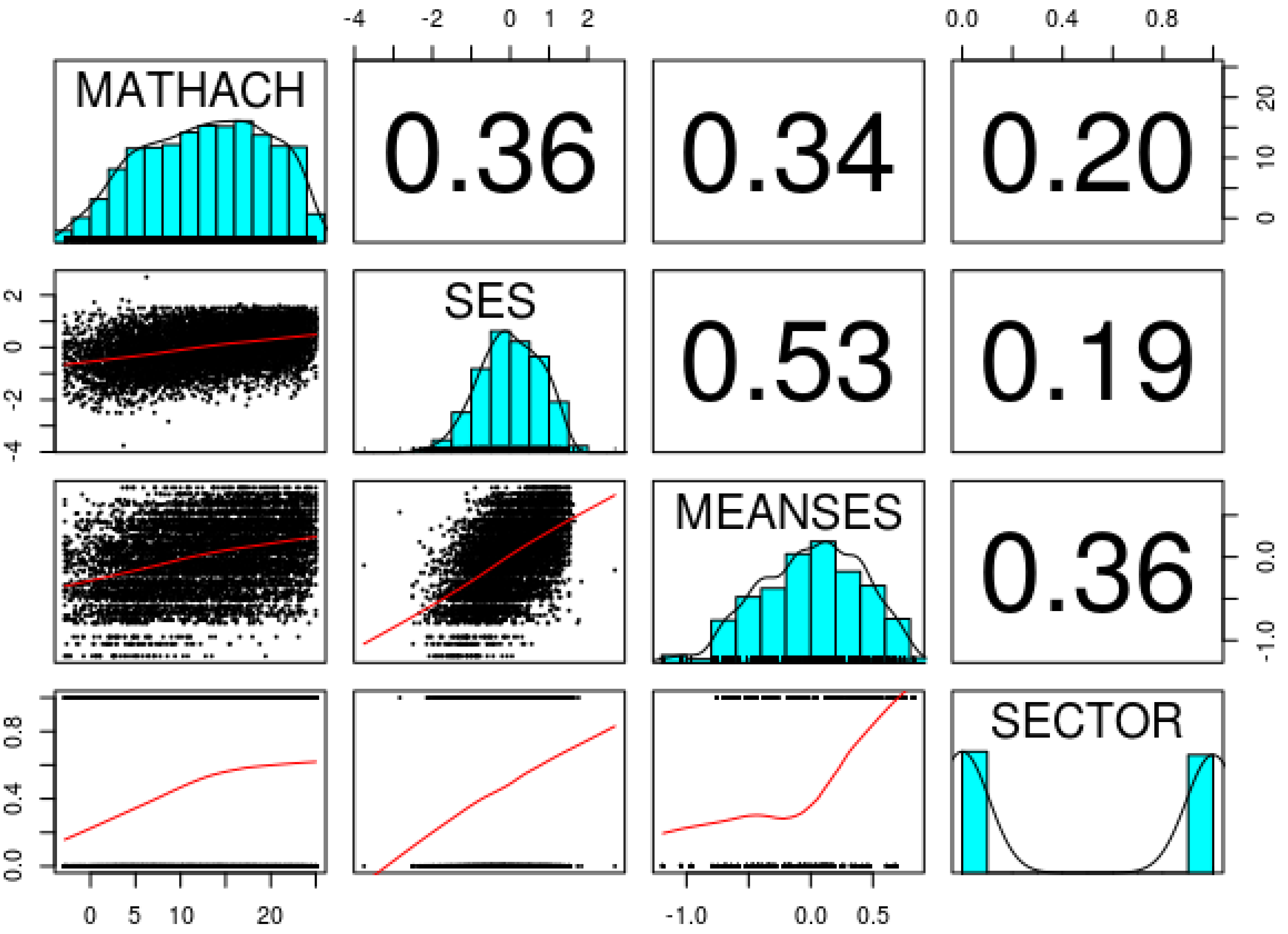
- Level 2: School
  - `size`: school size
  - `sector` (1 = Catholic, 0 = Public)
  - `pracad`: proportion in academic track
  - `disclim`: disciplinary climate
  - `himnty`: 1 = > 40% minority, 0 = < 40% minority
  - `meanses`: mean of Lv-1 SES

[1]: Check https://nces.ed.gov/surveys/hsb/ for more information

	ID	MINORITY	FEMALE	SES	MATHACH	SIZE	SECTOR	PRACAD	DISCLIM	HIMINTY	MEANSES
1	1224	0	1	-1.528	5.876	842	0	0.35	1.597	0	-0.428
2	1224	0	1	-0.588	19.708	842	0	0.35	1.597	0	-0.428
3	1224	0	0	-0.528	20.349	842	0	0.35	1.597	0	-0.428
4	1224	0	0	-0.668	8.781	842	0	0.35	1.597	0	-0.428
5	1224	0	0	-0.158	17.898	842	0	0.35	1.597	0	-0.428
6	1224	0	0	0.022	4.583	842	0	0.35	1.597	0	-0.428
7	1224	0	1	-0.618	-2.832	842	0	0.35	1.597	0	-0.428
8	1224	0	0	-0.998	0.523	842	0	0.35	1.597	0	-0.428
9	1224	0	1	-0.888	1.527	842	0	0.35	1.597	0	-0.428
10	1224	0	0	-0.458	21.521	842	0	0.35	1.597	0	-0.428

Student-level variables | **School-level variables**

	ID	MINORITY	FEMALE	SES	MATHACH	SIZE	SECTOR	PRACAD	DISCLIM	HIMINTY	MEANSES
996	2458	1	1	0.852	22.743	545	1	0.89	-1.484	1	0.234
997	2458	1	1	0.262	17.205	545	1	0.89	-1.484	1	0.234
998	2458	1	1	0.052	12.071	545	1	0.89	-1.484	1	0.234
999	2458	1	1	-0.468	19.161	545	1	0.89	-1.484	1	0.234
1000	2458	1	1	-0.268	12.332	545	1	0.89	-1.484	1	0.234
1001	2458	0	1	1.512	22.681	545	1	0.89	-1.484	1	0.234
1002	2458	1	1	0.182	4.928	545	1	0.89	-1.484	1	0.234
1003	2458	1	1	0.242	9.142	545	1	0.89	-1.484	1	0.234
1004	2458	0	1	1.072	24.488	545	1	0.89	-1.484	1	0.234
1005	2458	1	1	1.172	13.666	545	1	0.89	-1.484	1	0.234



MATHACH
SES
MEANSES
SECTOR
0.36
0.34
0.20
0.53
0.19
0.36
-4
-2
0
1
2
0.0
0.4
0.8
0
10
20
-1.0
0.0
0.5
0
5
10
20

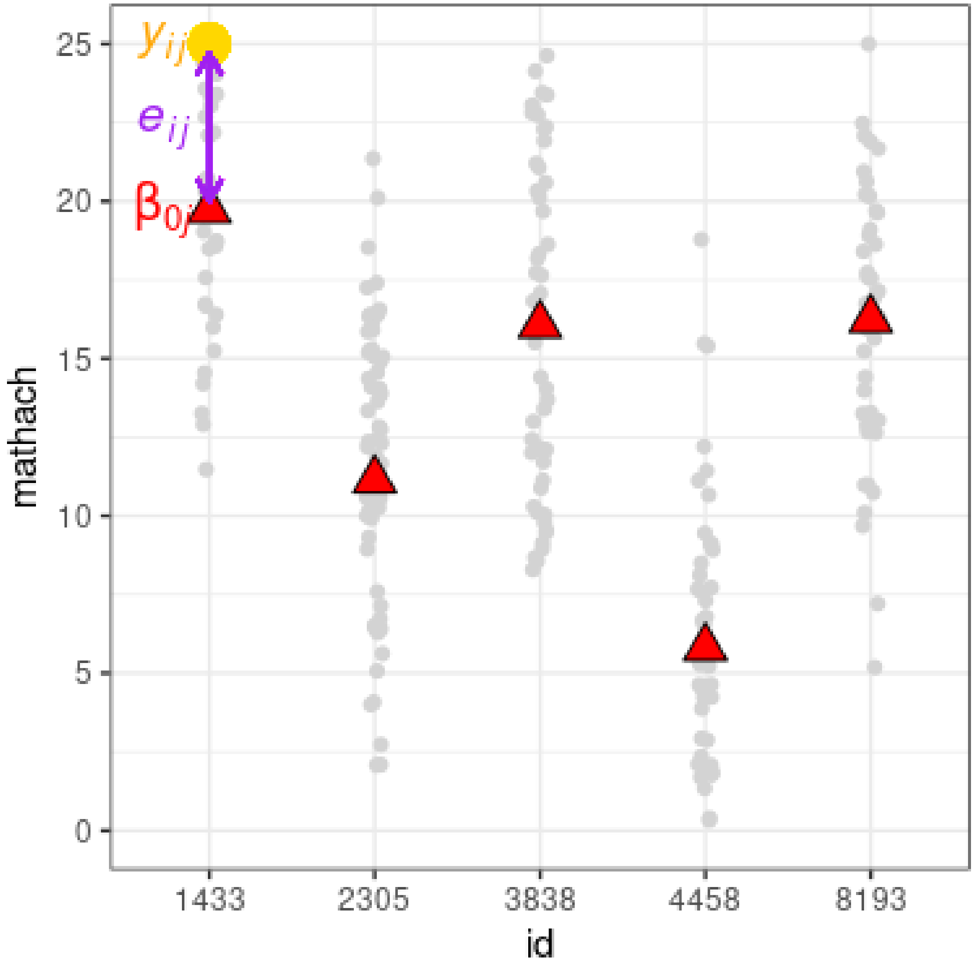
Research Questions

- Does math achievement vary across schools? How much is the variation?
- Do schools with higher mean SES have students with higher math achievement?

Random Intercept Model

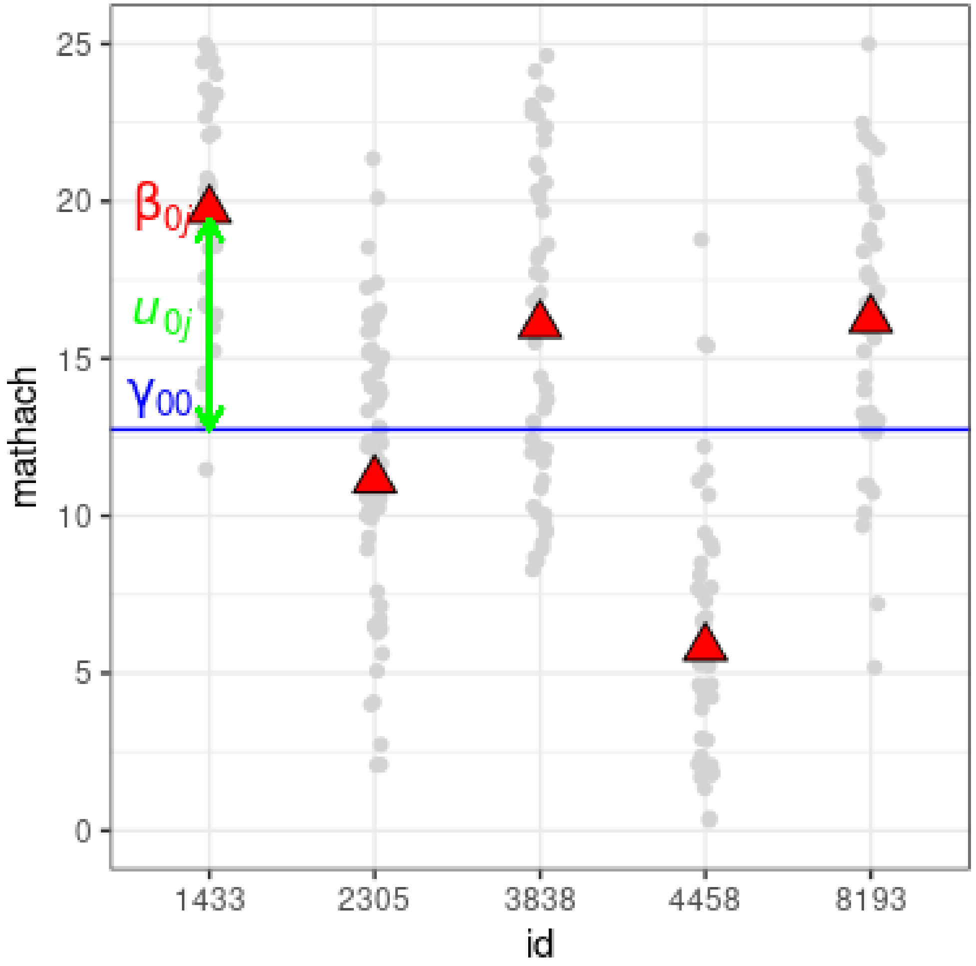
(Unconditional) Random Intercept Model

- Student level (Lv 1)
  - $\text{mathach}_{ij} = \beta_{0j} + e_{ij}$



(Unconditional) Random Intercept Model

- School level (Lv 2)
  - $\beta_{0j} = \gamma_{00} + u_{0j}$



(Unconditional) Random Intercept Model

- Student level (Lv 1)
  - $\text{mathach}_{ij} = \beta_{0j} + e_{ij}$
- School level (Lv 2)
  - $\beta_{0j} = \gamma_{00} + u_{0j}$

Combined:

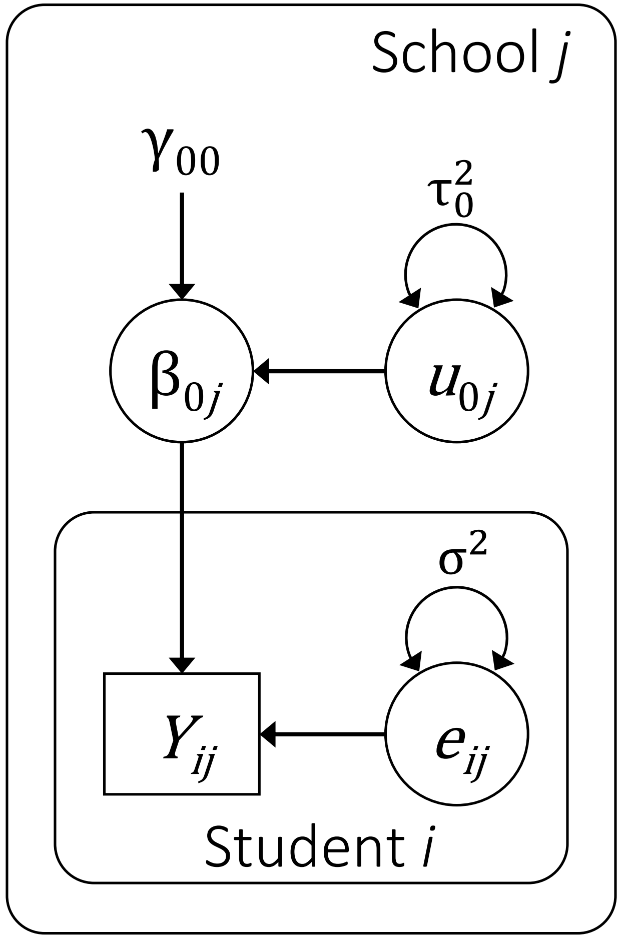
$$\text{mathach}_{ij} = \gamma_{00} + u_{0j} + e_{ij}$$

Score of student *i* in school *j*

= Grand mean (γ_{00}) + school deviation (u_{0j})
+ student deviation (e_{ij})

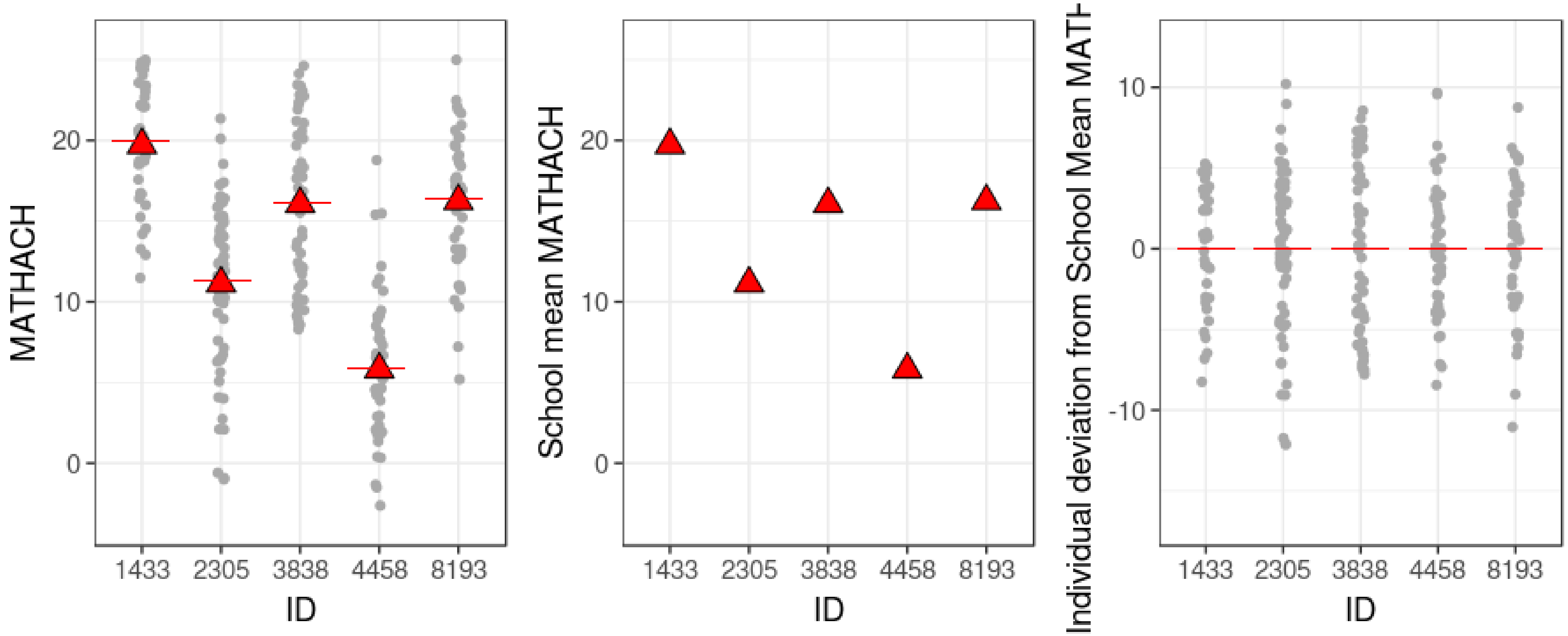
Model Diagram

- Student level (Lv 1)
  - $\text{mathach}_{ij} = \beta_{0j} + e_{ij}, \quad e_{ij} \sim N(0, \sigma)$
- School level (Lv 2)
  - $\beta_{0j} = \gamma_{00} + u_{0j}, \quad u_{0j} \sim N(0, \tau_0)$
- Combined:
  - $\text{mathach}_{ij} = \gamma_{00} + u_{0j} + e_{ij}$



Decomposing School- and Student-Level Information

- `mathach` = School info + Student info (Relative to School)



Terminology

- Fixed effects (γ): constant for everyone
- Random effects (e_{ij}, u_{0j}): varies for different observations/clusters
  - Describe by some probability distributions (e.g., normal)
  - Variance components: variance of random effects

Fixed Effects (R Output)

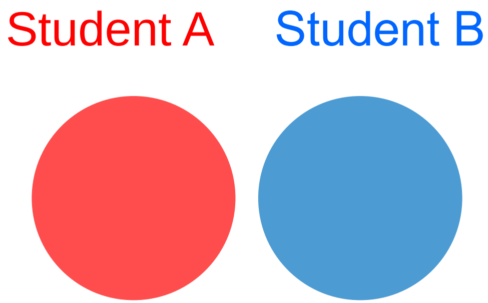
```
>#  Fixed effects:
>#               Estimate Std. Error t value
>#  (Intercept)  12.6370     0.2444   51.71
```

The estimated grand mean of MATHACH for all students is γ_{00} **= 12.64**, *SE* = **0.24**

Intraclass Correlation

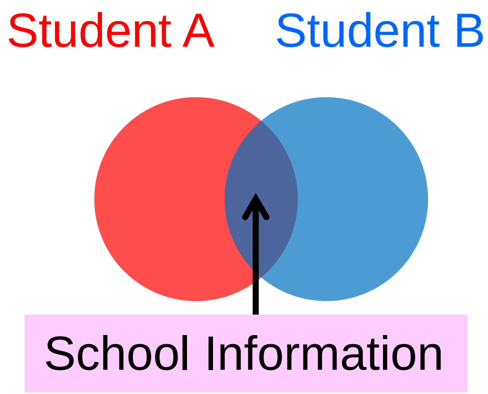
Intraclass Correlations (ICC; ρ)

- Independent



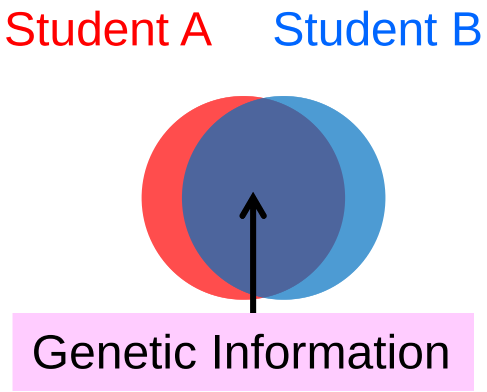
- ICC = 0

- Weakly Correlated



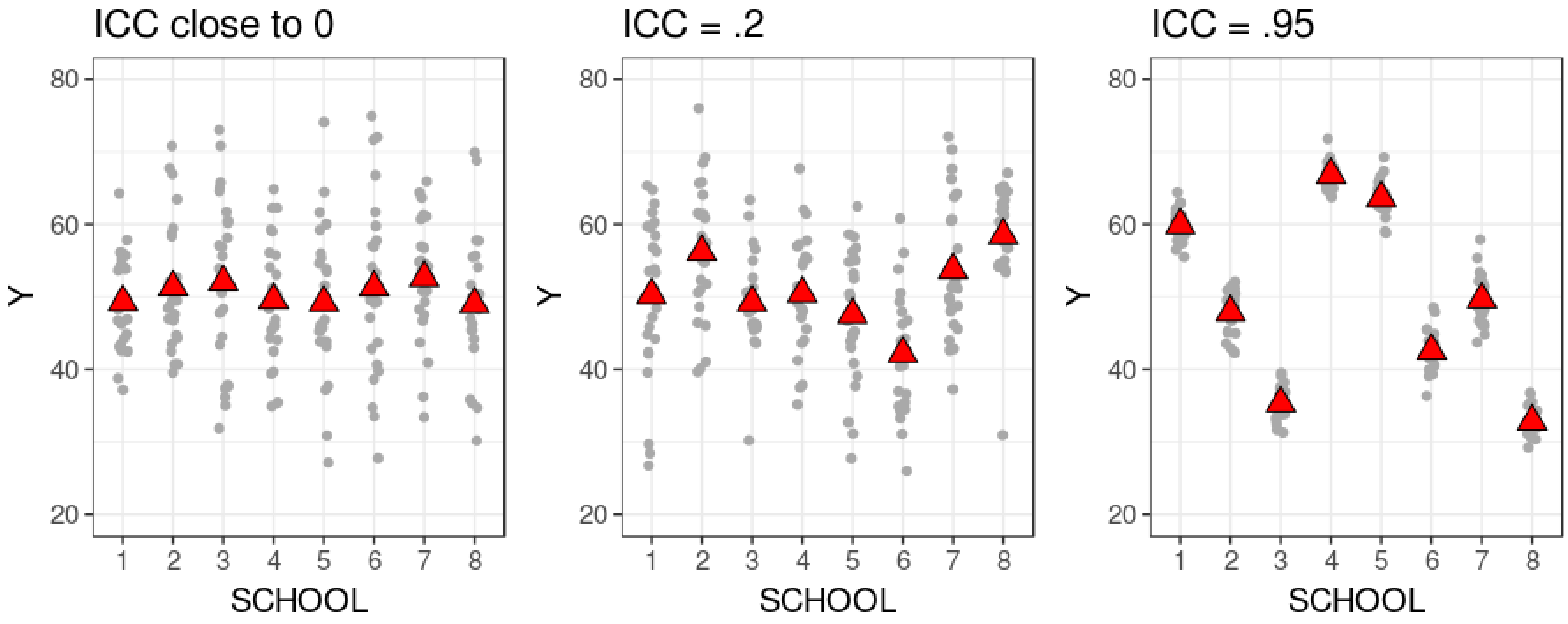
- ICC = .2

- Strongly Correlated



- ICC = .8

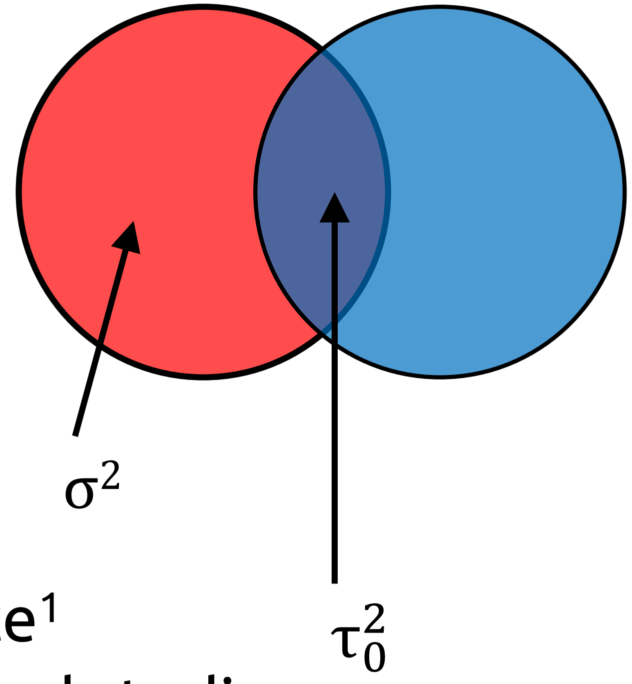
- ICC =
  1. Proportion of variance due to the higher (school-) level
  2. Average correlation between observations (students) in the same cluster (school)



Variance Components

- Var(u_{0j}) = τ_0^2 = between-school variance
- Var(e_{ij}) = σ^2 = within-school variance
- ICC:

$$\rho = \frac{\tau_0^2}{\tau_0^2 + \sigma^2}$$



  - Typical ICC = .1 to .25 for educational performance[1]
  - Higher ICCs for repeated measures and longitudinal studies

[1]: Hedges and Hedberg (2007), https://doi.org/10.3102/0162373707299706

R Output

```
># Random effects:
>#  Groups   Name        Variance Std.Dev.
>#  id       (Intercept)  8.614    2.935
>#  Residual             39.148    6.257
># Number of obs: 7185, groups:  id, 160
```

Variance of school means = 8.61
Variance of individual scores within a school = 39.15
ICC = 8.61 / (8.61 + 39.15) = **0.18**

Question: Does math achievement varies across schools? How much is the variation?

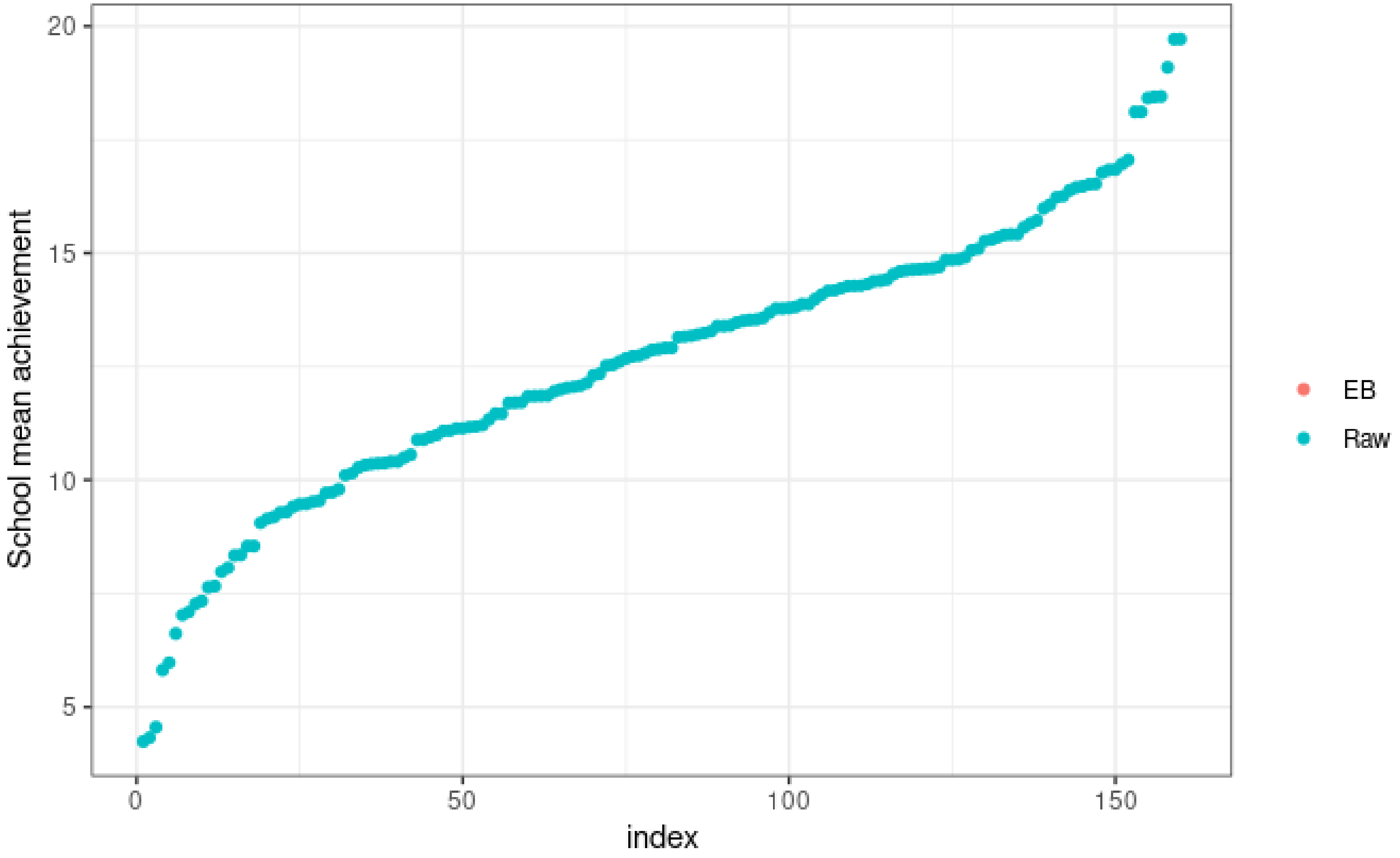
- Yes, there is evidence that student’s math achievement varies across schools.
- Variability at the school level accounts for 18% of the total variability of math achievement

Empirical Bayes Estimates

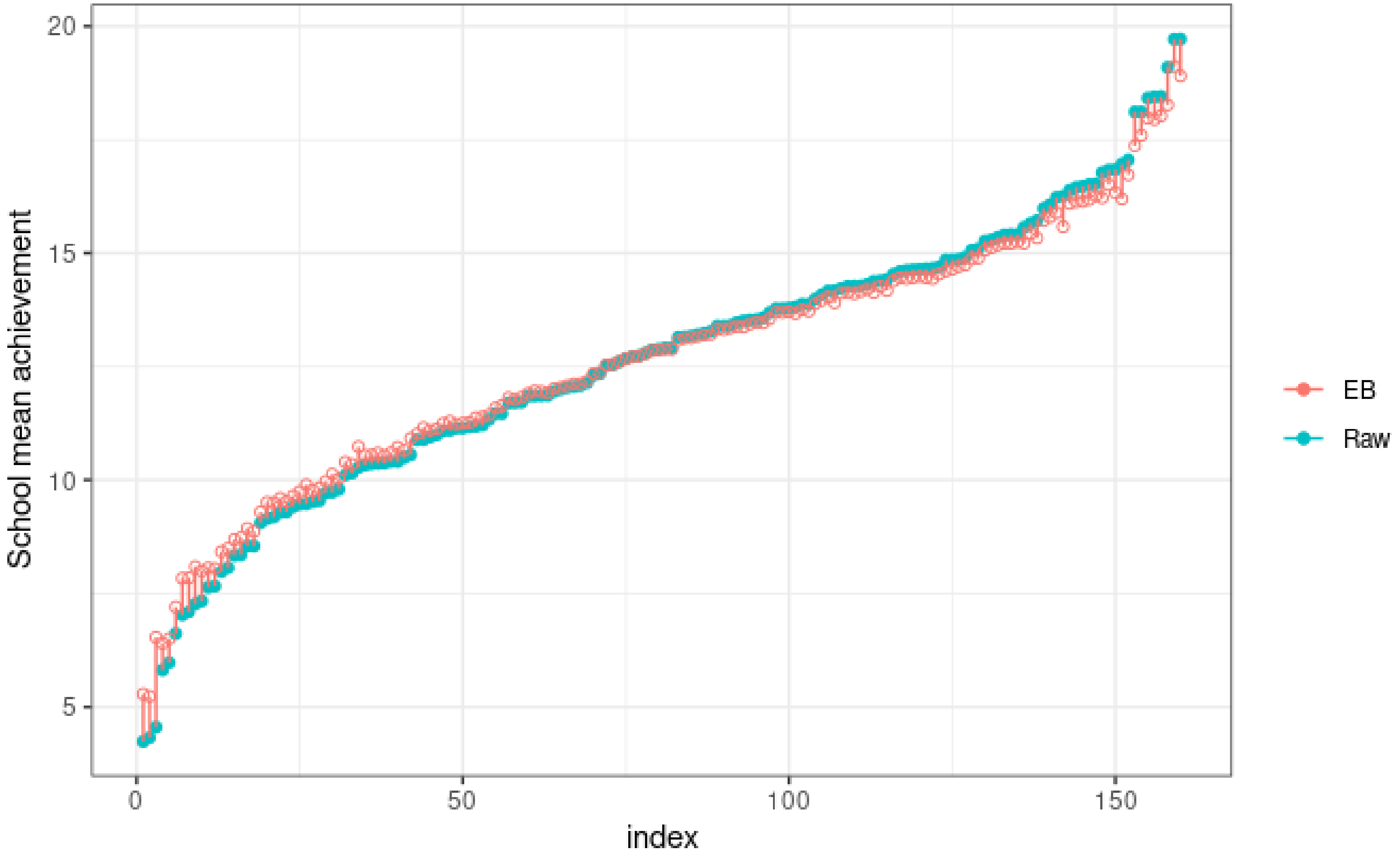
MLM Borrows Information

- β_{0j} = (population) mean math achievement of school *j*
- Most straightforward way to estimate β_{0j}:
  - Take the average of everyone in the sample in school *j*
- It may be unstable in small samples
- Instead, MLM borrows information from other schools

Also called *Shrinkage estimates*, *Best unbiased linear predictor* (BLUP), *Posterior modes*



Also called *Shrinkage estimates*, *Best unbiased linear predictor* (BLUP), *Posterior modes*



Empirical Bayes "Estimates"

$$\hat{\beta}_{0j}^{\mathrm{EB}} = \lambda_j \hat{\beta}_{0j}^{\mathrm{OLS}} + (1 - \lambda_j)\gamma_{00},$$

where

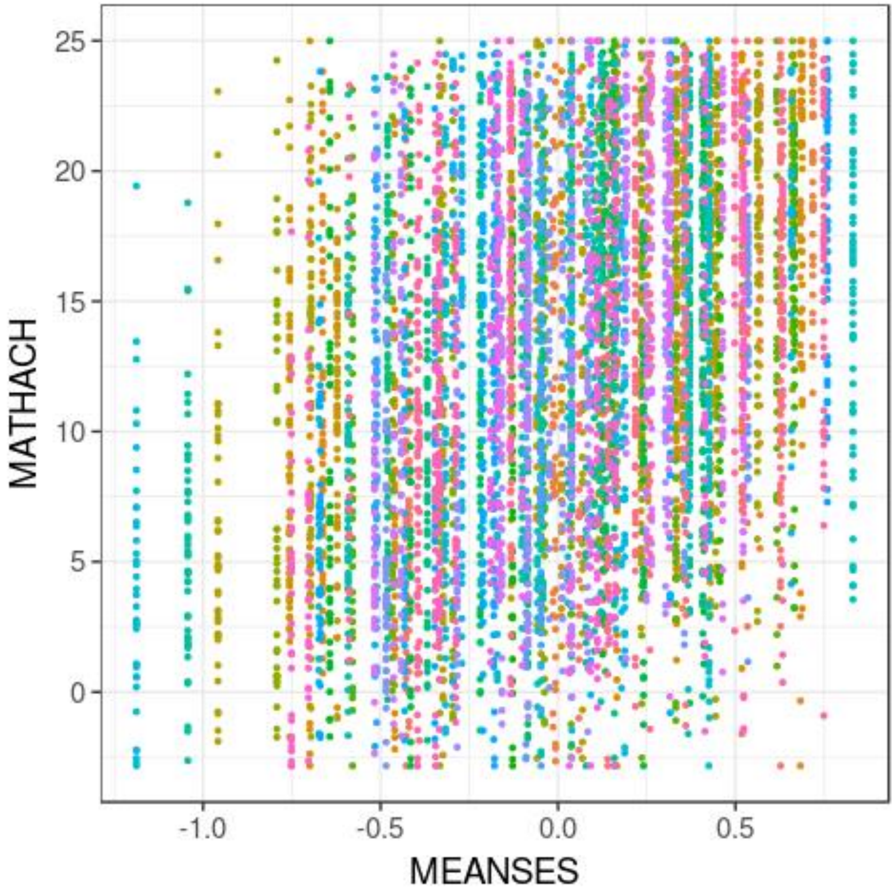
- $\lambda_j = \tau_0^2 / (\tau_0^2 + \sigma^2 / n_j)$ = reliability of group means
- In practice, the variance components need to be estimated

- Think: what happens when ICC = 0 (i.e., τ_0^2 = 0)? Or ICC = 1 (i.e., σ^2 = 0)?
- Read more on Snijders & Bosker, 4.8

Do schools with higher mean SES have students with higher math achievement?

Adding Predictors

- Why some schools have higher mean math achievement than others?



Why Not Simple Regression?

- `mathach` and `meanses` are at different levels
- Two (problematic) approaches:
  - Disaggregation (both variables as lv 1)
  - Aggregation (both variables as lv 2)

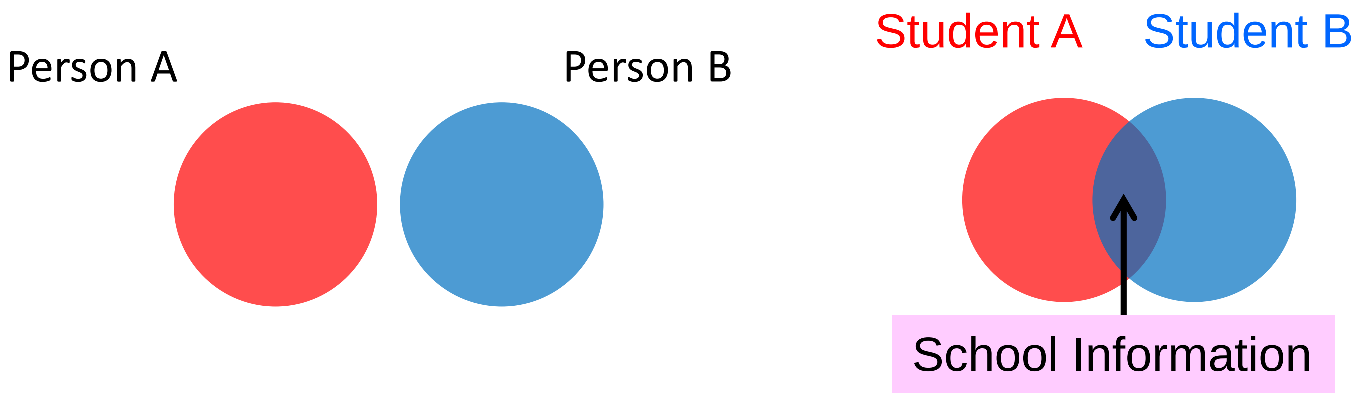
Problem of Disaggregation

"Miraculous multiplication of the number of units"
(Snijders & Bosker, p. 16)

- Only 160 schools, but regression uses $N = 7{,}185$

Dependent Observations

- Regression assumes *independent* observations

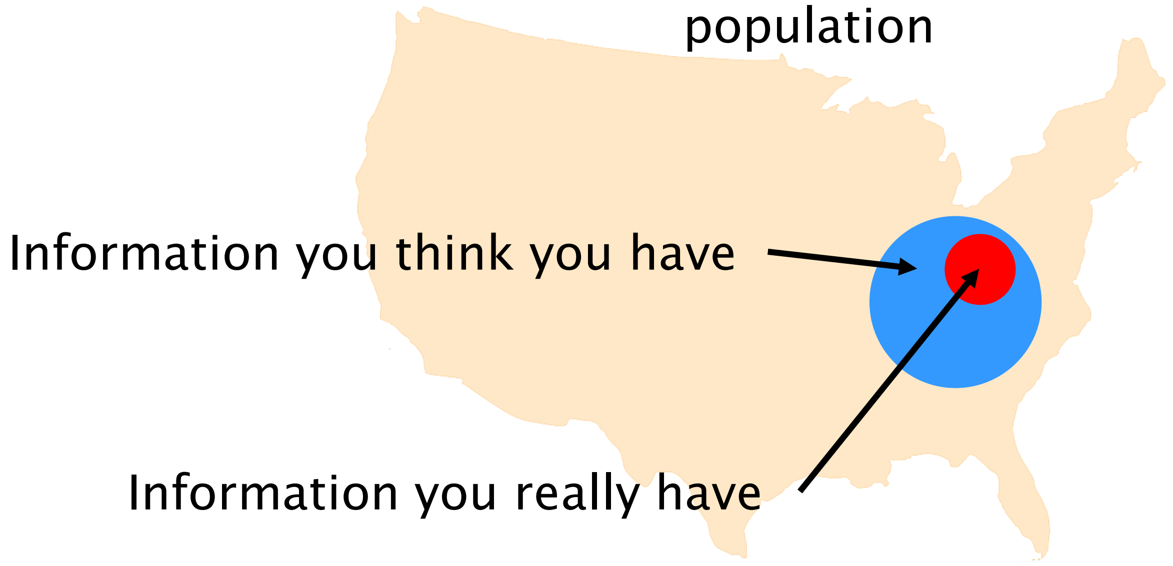


Design Effect

Design Effect (*Deff*)

- Dependent observations ➔ reduces information
  - Depends on overlap (ICC)

- *Deff* = 1 + (average cluster size – 1) × ICC
- $N_{eff} = N / Deff$



Underestimated Standard Error

- OLS on 7,185 students

```
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 12.71276    0.07622  166.80   <2e-16 ***
meanses      5.71680    0.18429   31.02   <2e-16 ***
```

- MLM

```
Fixed effects:
            Estimate Std. Error t value
(Intercept)  12.6494     0.1493   84.74
meanses       5.8635     0.3615   16.22
```

$$t = \frac{Est}{SE}$$

(Optional)
Approximate Standard Errors

- N = 7,185 students; J = 160 schools
- s^2_{meanses} = .170 = variance of MEANSES

```
Random effects:
 Groups   Name        Variance Std.Dev.
 id       (Intercept)  2.639   1.624
 Residual             39.157   6.258
Number of obs: 7185, groups:  id, 160
```

Approximate Standard Errors

- $SE_{\text{OLS}} \approx \sqrt{\frac{1}{s^2_{\text{MEANSES}}}\left(\frac{\tau_0^2+\sigma^2}{N}\right)} = \sqrt{\frac{1}{.170}\left(\frac{2.639+39.157}{7185}\right)} = .185$

τ_0^2 (lv-2) is divided by an incorrect sample size (lv-1)

- $SE_{\text{MLM}} \approx \sqrt{\frac{1}{s^2_{\text{MEANSES}}}\left(\frac{\tau_0^2}{J}+\frac{\sigma^2}{N}\right)} = \sqrt{\frac{1}{.170}\left(\frac{2.639}{160}+\frac{39.157}{7185}\right)} = .359$

Type I Error Inflation[1]

Cluster size	ICC	*Deff*	Type I Error	Cluster size	ICC	*Deff*	Type I Error
10	0	1.00	.05	10	.20	2.80	.28
25	0	1.00	.05	25	.20	5.80	.46
100	0	1.00	.05	100	.20	20.80	.70
10	.05	1.45	.11	10	.40	5.50	.46
25	.05	2.20	.19	25	.40	13.00	.63
100	.05	5.95	.43	100			

For the HSB data, *Deff* = ??

- Lai & Kwok (2015):[2] MLM needed when *Deff* > 1.1

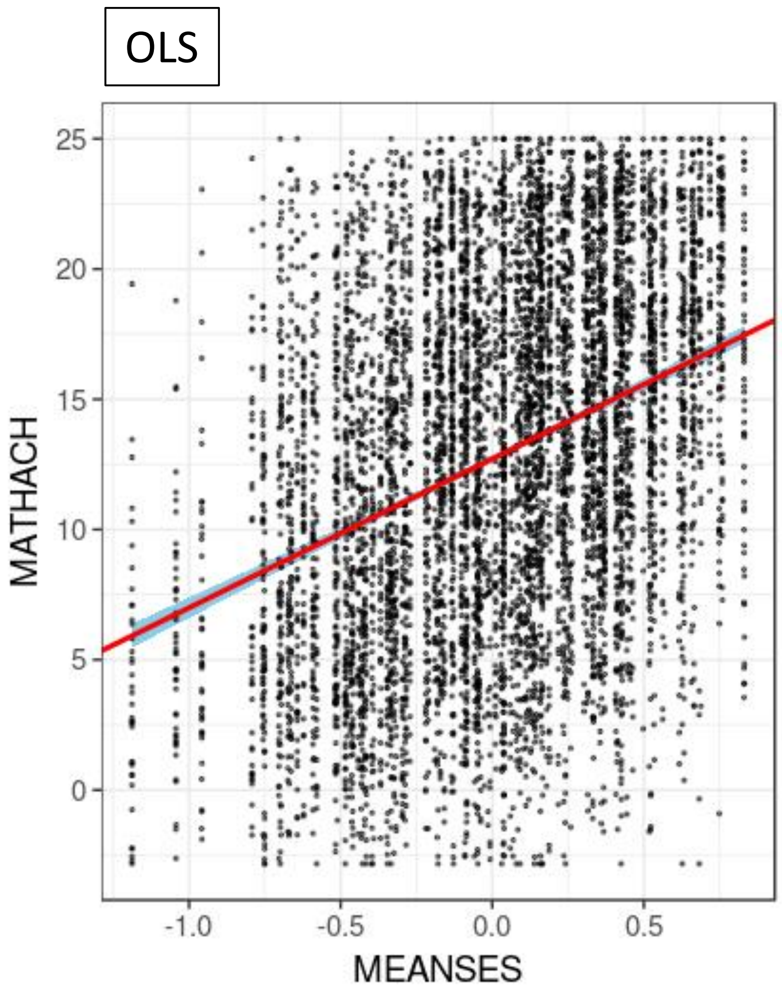
[1]: Table adapted from Barcikowski (1983) [2]: https://doi.org/10.1080/00220973.2014.907229

Exercise

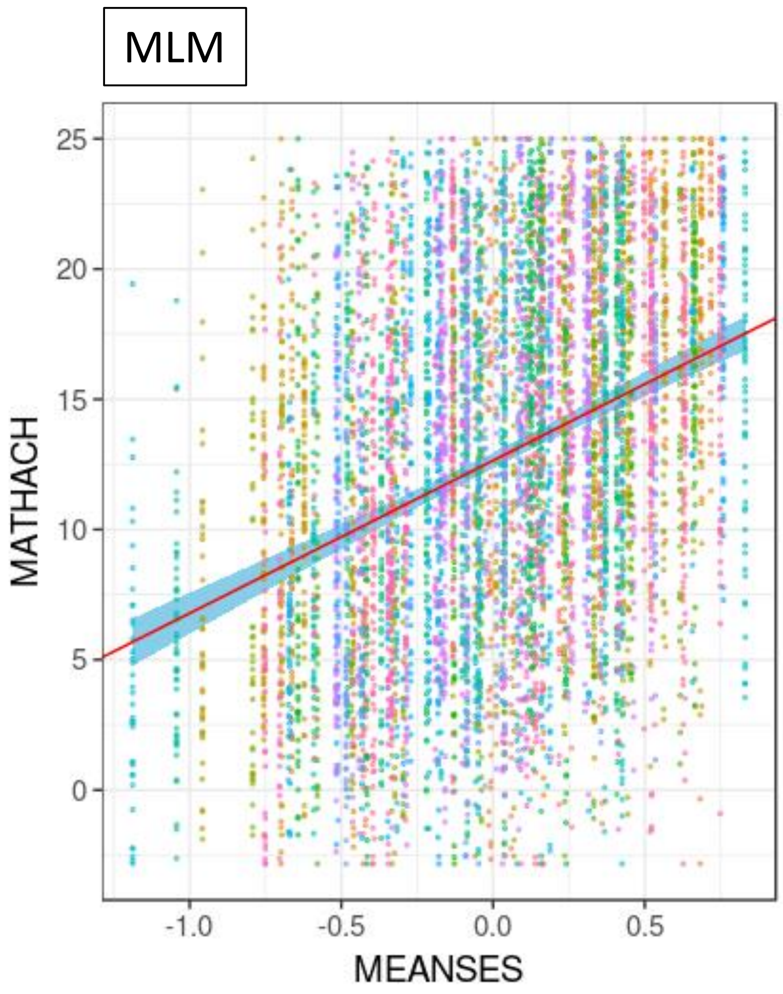
- *Deff* = 1 + (average cluster size – 1) × ICC
- Average cluster size = 7,185 / 160 ≈ 44.91
- ICC = 0.18

- Bonus Challenge: What is the design effect for a longitudinal study of 5 waves with 30 individuals, and the ICC for the outcome is 0.5?

Overconfidence (Disaggregation)



95 % CI of slope = [5.36, 6.08]



95 % CI of slope = **[5.16, 6.57]**

Problem of Aggregation

- Student-level information is ignored
- OLS on 160 schools

```
             Estimate Std. Error t value Pr(>|t|)
(Intercept)   12.6219     0.1533   82.35   <2e-16 ***
MEANSES        5.9093     0.3714   15.91   <2e-16 ***
```

- MLM

```
Fixed effects:
             Estimate Std. Error t value
(Intercept)   12.6494     0.1493   84.74
MEANSES        5.8635     0.3615   16.22
```

SE is slightly overestimated

Model Equations

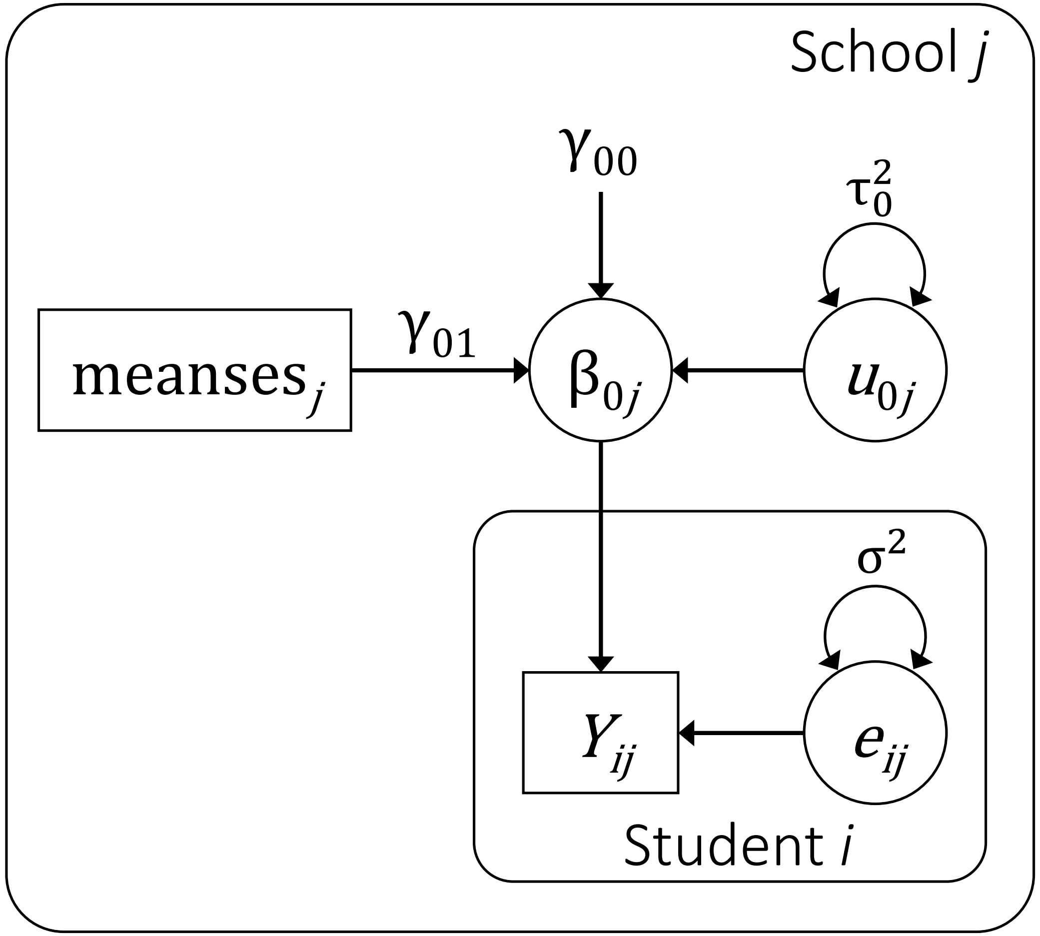
- Lv 1: $\text{mathach}_{ij} = \beta_{0j} + e_{ij}$

- Lv 2: $\beta_{0j} = \gamma_{00} + \gamma_{01}\ \text{meanses}_j + u_{0j}$

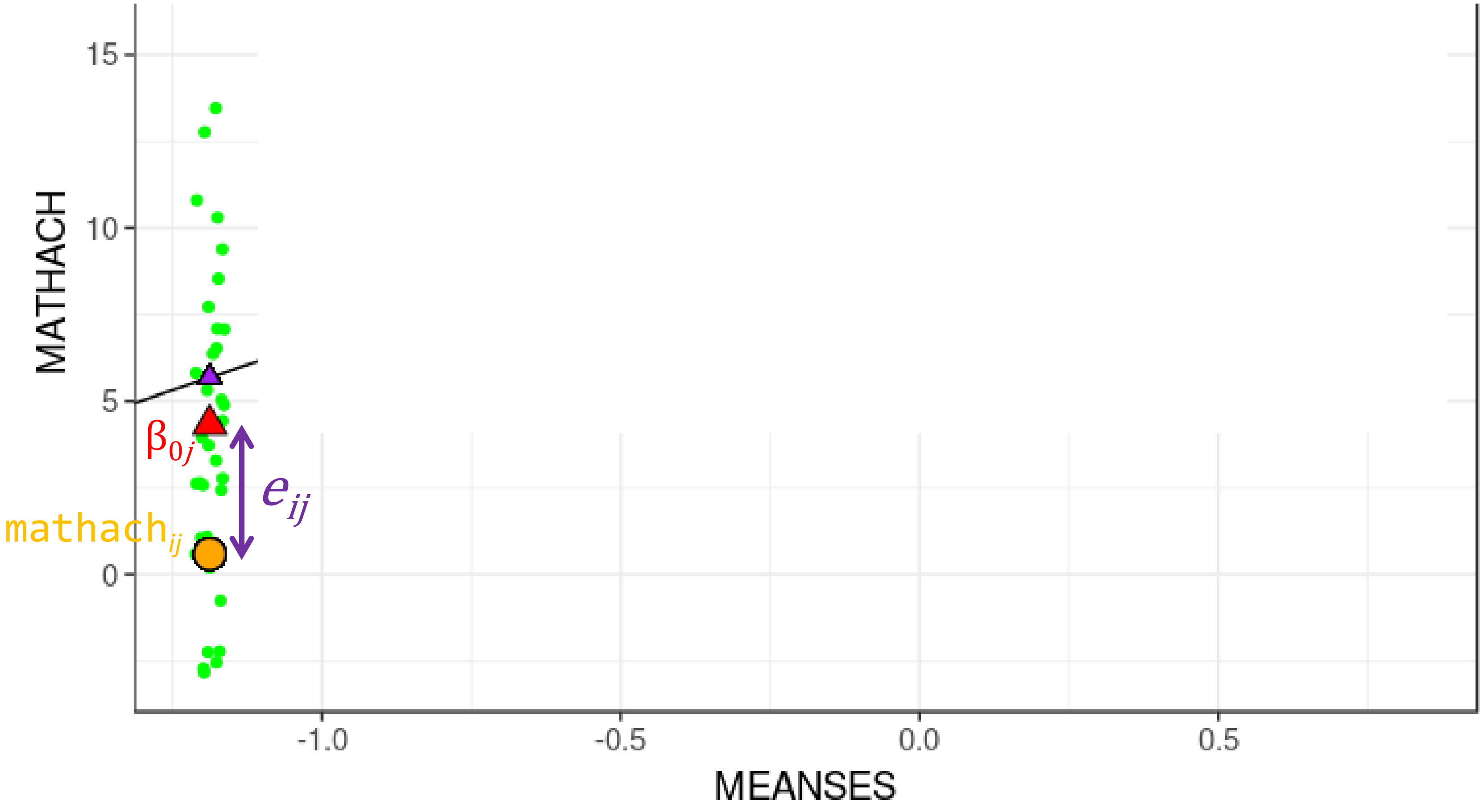
- Combined: $\text{mathach}_{ij} = \gamma_{00} + \gamma_{01}\ \text{meanses}_j + u_{0j} + e_{ij}$

Model Equations

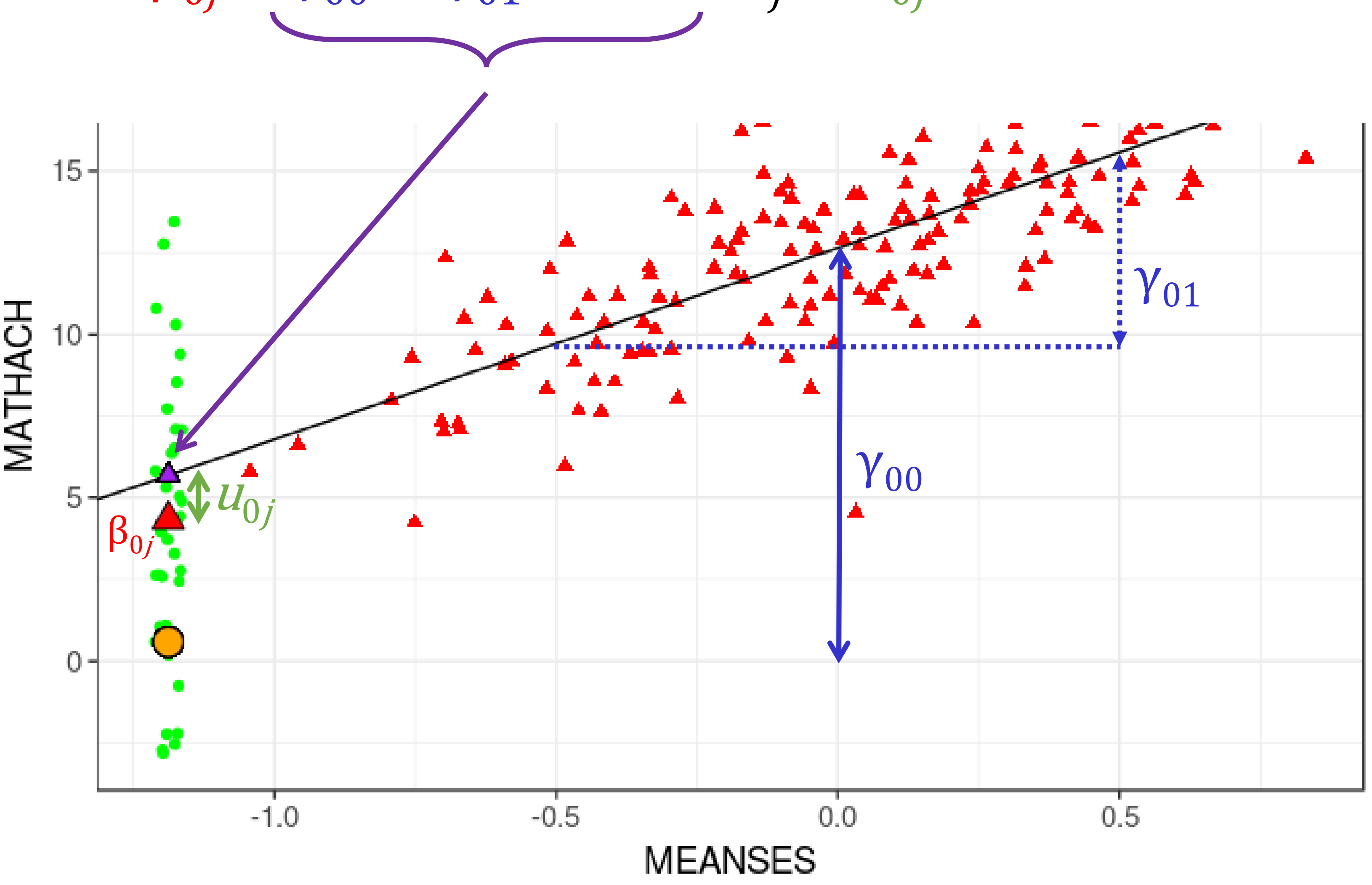
- Lv 1: $mathach_{ij} = \beta_{0j} + e_{ij}$
  $e_{ij} \sim N(0, \sigma)$
- Lv 2: $\beta_{0j} = \gamma_{00} + \gamma_{01}\ meanses_j + u_{0j}$
  $u_{0j} \sim N(0, \tau_0)$
- Combined:
  $mathach_{ij} = \gamma_{00} + \gamma_{01}\ meanses_j + u_{0j} + e_{ij}$



Lv 1: $mathach_{ij} = \beta_{0j} + e_{ij}$



Lv 2: $\beta_{0j} = \gamma_{00} + \gamma_{01}\ \text{meanses}_j + u_{0j}$



Run the Model in R

```
Fixed effects:
            Estimate Std. Error t value
(Intercept)  12.6494     0.1493   84.74
meanses       5.8635     0.3615   16.22
```

The estimated school mean of mathach when meanses = 0 is γ_{00} = **12.65 (*SE* = 0.15)**

The model predicts that students from two schools with 1 unit difference in meanses will have an average difference of γ_{01} = **5.86 (*SE* = 0.36)** units in mathach

Run the Model in R

```
Random effects:
 Groups   Name        Variance Std.Dev.
 id       (Intercept)  2.639   1.624
 Residual             39.157   6.258
Number of obs: 7185, groups:  id, 160
```

Variance of deviations of school means from the regression line = Var(u_{0j}) = 2.64

Variance of individual scores within a school = Var(e_{ij}) = 39.16

Statistical Inferences

- It's important to understand that the coefficients you obtained in software are merely *estimates*, which involves *uncertainty*
- Confidence intervals
  - Wald intervals
  - Likelihood-based intervals
- Hypothesis testing (to be discussed later)

Confidence Intervals (Wald)

- 95% CI for γ_{01} = 5.86 ± 2 × 0.36 = [5.16, 6.57]
  - Can be obtained in most software

At 95% confidence level, one unit difference in school-level MEANSES is associated with an average difference in MATHACH of **5.16** to **6.57** units

Confidence Intervals (Likelihood-Based)

```
> confint(m_lv2, parm = "beta_")
Computing profile confidence intervals ...
                2.5 %    97.5 %
(Intercept) 12.356615 12.941707
meanses      5.155769  6.572415
```

- Easily obtained in the R package `lme4`
- Usually more accurate than Wald intervals, especially with smaller sample sizes
- With a large sample size, the difference is minimal